

Calibrated Fibrations

Edward Goldstein

February 1, 2008

Abstract

In this paper we investigate the geometry of Calibrated submanifolds and study relations between their moduli-space and geometry of the ambient manifold. In particular for a Calabi-Yau manifold we define Special Lagrangian submanifolds for any Kahler metric on it. We show that for a choice of Kahler metric the Borcea-Voisin threefold has a fibration structure with generic fiber being a Special Lagrangian torus. Moreover we construct a mirror to this fibration. Also for any closed G_2 form on a 7-manifold we study coassociative submanifolds and we show that one example of a G_2 manifold constructed by Joyce in [10] is a fibration with generic fiber being a coassociative 4-torus. Similarly we construct a mirror to this fibration.

1 Introduction

In their seminal paper [7] Harvey and Lawson defined the notion of calibrations. Let M be a Riemannian manifold and φ be a closed k -form. We say that φ is a calibration if for any k -dimensional plane κ in the tangent bundle of M , we have $\varphi|_{\kappa} \leq \text{vol}(\kappa)$. We call a k -dimensional submanifold L a calibrated submanifold if $\varphi|_L = \text{vol}(L)$. It is easy to see that calibrated submanifolds minimize volume in their homology class and thus provide examples of minimal varieties. For a thorough discussion and numerous examples we refer the reader to [6],[7],[14].

In this paper we study the geometry of Calibrated submanifolds and investigate relations between their moduli-space and geometry of the ambient manifold. The paper is organized as follows:

In section 2 we prove several comparison theorems for the volume of small balls in a Calibrated submanifold of a Riemannian manifold M , whose sectional curvature is bounded from above by some K . Let L be a minimal submanifold in M , $p \in L$ a point and $B(p, r)$ is a ball of radius r around p in M . Then there are a number of results on comparison between the volume of $L \cap B(p, r)$ and a volume of a ball of radius r in a space form of constant curvature K (see [1], [11], [16]). Our main result in section 2 is Theorem 2.0.2, which states that if L is calibrated then the volume of a ball of radius r in the induced metric on L (which is smaller than $L \cap B(p, r)$) is greater than the volume of a ball of the same radius in a space form of constant curvature K for $r \leq r_0$ with

r_0 depending only on the ambient manifold M . As a corollary we deduce that there is an upper bound on a diameter of a calibrated submanifold in a given homology class.

In section 3 we investigate Special Lagrangian Geometry on a Calabi-Yau manifold. In section 3.1 we define Special Lagrangian submanifolds for any choice of Kahler metric on a Calabi-Yau manifold and give basic facts pertaining to Special Lagrangian (SLag) Geometry. We will also prove a result about finite group actions on Calabi-Yau manifolds and construct several new examples of SLag submanifolds.

In section 3.2 we study connections between moduli-space of Special Lagrangian submanifolds and global geometry of the ambient Calabi-Yau manifold. We will be interested in submanifolds, which satisfy condition \star on their cohomology ring (defined in section 3.2). In particular tori satisfy this condition. We state 2 conditions on an ambient manifold for each of those the moduli-space is not compact. These conditions hold in many examples, and so we got a non-compactness theorem for the moduli-space.

In section 3.3 we use results of 2 previous sections to investigate a Borcea-Voisin threefold in detail. We find a Kahler metric on it for which we can completely characterize singular Special Lagrangian submanifolds (they will be a product of a circle with a cusp curve). Moreover SLag submanifolds don't intersect and the compactified moduli-space fills the whole Calabi-Yau manifold, i.e. Borcea-Voisin 3-fold fibers with generic fiber being a Special Lagrangian torus. We also construct a mirror to this fibration. Thus the SYZ conjecture (see [22]) holds in this example.

In section 3.4 we will examine holomorphic functions on a Calabi-Yau manifold in a neighbourhood of a Special Lagrangian submanifold. An immediate consequence of the fact that SLag submanifolds are 'Special' is Theorem 3.4.1, which states that the integral of a holomorphic function over SLag submanifolds is a constant function on their moduli-space. This will give a restriction on how a family of SLag submanifolds might approach a singularity (Corollary 3.4.1) and also will give a restriction on SLag submanifolds asymptotic to a cone in \mathbb{C}^n (Theorem 3.4.2).

In section 4 we study coassociative submanifolds on G_2 manifolds. We extend a coassociative condition for any choice of a closed (but not necessarily co-closed) G_2 form. Deformation of coassociative submanifolds will still be unobstructed and the moduli-space is smooth of dimension $b_2^+(L)$, where L is a coassociative submanifold. We will show that one example of a G_2 manifold constructed by Joyce in [10] is a fibration with generic fiber being a coassociative 4-torus. We also construct a mirror to this fibration.

There are a number of natural questions that arise from this paper. One is to give a systematic way to construct fibrations on resolutions of torus quotients (both for SLag and coassociative geometry). Another point is that we produced those fibrations for certain special choices of structures on the ambient manifold (a certain choice of Kahler metric or a certain closed G_2 form). We would like to get fibrations for any other isotopic structure. If we have a 1-parameter family of structures then we obtain a 1-parameter family of moduli-spaces Φ_t . Suppose

that Φ_0 compactifies to a fibration of the ambient manifold. We conjecture that so does each Φ_t (both for SLAG and coassociative geometries). This in particular would imply the existence of SLAG fibration for the Calabi-Yau metric on a Borcea-Voisin threefold and coassociative fibration for a parallel G_2 structure. We hope to address those issues in a future paper.

Acknowledgments : This paper is written towards author's Ph.D. at the Massachusetts Institute of Technology. The author wants to thank his advisor, Tom Mrowka, for initiating him into the subject and for constant encouragement. He is also grateful to Gang Tian for a number of useful conversations. Special thanks go to Grisha Mikhalkin for explaining the Viro construction in Real Algebraic Geometry, which was used to construct examples of real quintics.

2 Volume Comparison for Calibrated Submanifolds

If a Riemannian manifold M has an upper bound K on its sectional curvature then the volume of a sufficiently small ball in M is greater than the volume of a ball of the same radius in a space form of curvature K . It turns out that this holds more generally for calibrated submanifolds of M . Namely we have a following theorem :

Theorem 2.0.1 : *Let φ be a calibrating k -form on an ambient manifold M and L be a calibrated submanifold. Let the sectional curvature of M be bounded from above by K . Let $r \leq \min(\text{inrad}(M), \frac{\pi}{\sqrt{K}})$. Let $p \in L$ and $B(p, r)$ be a ball of radius r around p in M and $B^K(r)$ be a ball of radius r in a k -dimensional space of constant sectional curvature K . Then*

$$\text{vol}(L \cap B(p, r)) \geq \text{vol}(B^K(r))$$

Remark : If φ is a volume form on M , then this is Gunther's volume comparison theorem.

Proof of Theorem 2.0.1: The proof is based on a following Lemma, which is a counterpart to Rauch comparison theorem :

Lemma 2.0.1 : *Let M be a (complete) Riemannian manifold whose sectional curvature is bounded from above by K and $\gamma : [0, t] \mapsto M$ be a unit speed geodesic. Let Y be a Jacobi field along γ which vanishes at 0, orthogonal to γ' and $t \leq \frac{\pi}{\sqrt{K}}$. Then its length $|Y(\theta)|$ satisfies the following differential inequality $|Y|'' + K|Y| \geq 0$.*

Moreover if a function Ψ is a solution to $\Psi'' + K \cdot \Psi = 0$, $\Psi(0) = 0$ and $\Psi(t) = |Y(t)|$ then

$$\Psi(\theta) \geq |Y(\theta)|$$

for $0 \leq \theta \leq t$

Proof :

First a condition on t means that Y doesn't vanish on $(0, t]$ by Rauch Comparison theorem. We have $|Y| = \sqrt{\langle Y, Y \rangle}$, $|Y|' = \frac{\langle \nabla_t Y, Y \rangle}{|Y|}$,

$$|Y|'' = \frac{|\nabla_t Y|^2 - \langle Y, R(\gamma', Y)\gamma' \rangle}{|Y|^3} - \frac{\langle \nabla_t Y, Y \rangle^2}{|Y|^3} \geq \frac{|\nabla_t Y|^2 |Y|^2 - \langle \nabla_t Y, Y \rangle^2}{|Y|^3} - K|Y| \geq -K|Y|$$

by Cauchy-Schwartz inequality. Here R is a curvature operator, γ' is a (unit length) tangent field to geodesic γ . Since Y is orthogonal to γ' then $\frac{\langle R(\gamma', Y)\gamma', Y \rangle}{|Y|^2}$ is the sectional curvature of a plane through Y and γ' , which is less than K .

For the second claim consider $F = \frac{|Y|}{\Psi}$. Ψ is positive on the interval $(0, t]$ and hence F is well defined on that interval. Also $F' = \frac{|Y|'\Psi - \Psi'|Y|}{\Psi^2}$. Consider $G = |Y|'\Psi - \Psi'|Y|$. $G(0) = 0$, $G' = |Y|''\Psi - \Psi''|Y| \geq 0$. So $G \geq 0$, i.e. $F' \geq 0$. Now $F(t) = 1$, so $F \leq 1$ i.e. $|Y| \leq F$ Q.E.D.

Now we can prove **Theorem 2.0.1:**

Let d_p be a distance function to p on M . Then for an open dense set of full measure of values t , t is a regular value of d_p restricted to L . Let now

$$f(t) = \text{vol}(L \cap B(p, t)) \text{ and } g(t) = \int_{L \cap B(p, t)} |\nabla_L d_p|$$

We also consider an analogous situation on \bar{L} - a space form of constant curvature K . Then $\bar{f} = \bar{g}$ because $|\nabla \bar{d}_p| = 1$ on \bar{L} .

For t a regular value as above we have by the co-area formula :

$$f'(t) \geq g'(t), \quad g'(t) = \text{vol}(S_t) \quad (1)$$

and

$$\bar{f}' = \text{vol}(\bar{S}_t), \quad (2)$$

here $S_t = d_p^{-1}(t) \cap L$.

Consider now a map $\xi : S_t \times [0, t] \mapsto M$, $\xi(a, \theta) = \exp(\frac{\theta}{t} \exp^{-1}(a))$, here $a \in S_t$, $\theta \in [0, t]$. Then $\text{vol}(\xi(S_t \times [0, t])) \geq f(t)$.

Indeed let ρ be a $k-1$ form on $B(p, r)$ s.t. $d\rho = \varphi$ (such ρ exists by Poincare Lemma). Then by the calibrating condition

$$\text{vol}(\xi(S_t \times [0, t])) \geq \int_{S_t \times [0, t]} \xi^* \varphi = \int_{S_t} \rho = \int_{B(p, t) \cap L} \varphi = f(t)$$

Also on \bar{L} we have $\bar{f}(t) = \text{vol}(\xi(\bar{S}_t \times [0, t]))$.

We need to estimate $h(t) = \text{vol}(\xi(S_t \times [0, t]))$. Let g' be the product metric on $S_t \times [0, t]$. Then $h(t) = \int_{S_t \times [0, t]} \text{Jac}(d\xi) dg'$.

To estimate $\text{Jac}(d\xi)$ at point (a, θ) we take an o.n. basis $v_1 \dots v_{k-1}$ to S_t at a . Then $d\xi(v_i)$ is a value of a Jacobi field along a unit speed geodesic $(\exp(s \cdot \frac{\exp^{-1}(a)}{t})|_{s \in [0, \infty)})$ at $s = \theta$ which is orthogonal to this geodesic,

vanishes at 0 and those length is 1 at $s = t$.
Let $F_t(\theta)$ solve $F_t'' + K \cdot F_t = 0$, $F_t(0) = 0$, $F_t(t) = 1$.
By Lemma 2.0.1 we have $|d\xi(v_i)| \leq F_t(\theta)$, so

$$Jac(d\xi) \leq (F_t(\theta))^{k-1} \quad (3)$$

We can consider an analogous situation on \overline{L} and in that case we have an equality $Jac(d\overline{\xi}) = (F_t(\theta))^{k-1}$. So

$$\overline{f(t)} = \int_{\overline{S_t} \times [0, t]} (F_t(\theta))^{k-1} = vol(\overline{S_t}) \cdot \int_{[0, t]} (F_t(\theta))^{k-1} d\theta = (by(2)) = \overline{f'}(t) \cdot \alpha(t)$$

(here $\alpha(t) = \int_{[0, t]} (F_t(\theta))^{k-1} d\theta$).

Returning now to our calibrated submanifold we deduce from (3) and (1) that $f(t) \leq f'(t) \cdot \alpha(t)$.

So $\frac{f'(t)}{f(t)} \geq \frac{\overline{f'}(t)}{\overline{f}(t)}$, i.e. $\ln(f)' \geq (\ln(\overline{f}) - \epsilon)'$ for any $\epsilon > 0$. Having ϵ fixed we can choose t_0 small enough s.t. $\ln f(t_0) \geq \ln \overline{f}(t_0) - \epsilon$.

Now $\ln f(\theta)$ is defined for a.e. θ and is an increasing function on θ , so

$$\ln f(t) \geq \ln f(t_0) + \int_{[t_0, t]} \ln f' \geq \ln \overline{f}(t_0) - \epsilon + \int_{[t_0, t]} (\ln \overline{f})' = \ln \overline{f}(t) - \epsilon$$

Now ϵ was arbitrary, hence $\ln f(t) \geq \ln \overline{f}(t)$ i.e. $f(t) \geq \overline{f}(t)$ Q.E.D .

We wish to discuss the compactification of some moduli-space of calibrated submanifolds in a given homology class. If we have a moduli-space Φ we can look on it as a subspace in the space of rectifiable currents. k -dimensional currents have a mass norm \mathbf{M} and a flat norm \mathcal{F} (see [18], p.42)

$$\mathbf{M}(L) = \sup \left(\int_L \eta \mid \eta \text{ a } k\text{-form}, \forall p \in M : |\eta(p)| \leq 1 \right)$$

$$\mathcal{F}(A) = \inf(\mathbf{M}(A) + \mathbf{M}(B) \mid L = A + \partial B)$$

Since all the submanifolds in Φ are closed and have the same volume, then by the Fundamental compactness theorem (theorem 5.5 in [18]) we have that the closure $\overline{\Phi}$ of Φ in the flat topology is compact.

Also for compact subsets of M there is a Gromov-Hausdorff distance function d^{GH} , there

$$d^{GH}(K, N) = \sup_{p \in K} \inf_{q \in N} d(p, q)$$

Using Theorem 2.0.1 we get

Corollary 2.0.1 : *There is a constant $C = C(M, \varphi)$ s.t. for $K, N \in \Phi$ we have $d^{GH}(K, N) \leq C \cdot (\mathcal{F}(K - N))^{\frac{1}{k+1}}$.*

Proof : Suppose $d^{GH}(K, N) = r$. Then we have a point $p \in K$ s.t. $d(p, N) = r$. It is easy to construct a nonnegative function f supported in a ball $B(p, r)$ which

is equal to 1 on a ball $B(p, r/2)$ and s.t. $|\nabla(f)| \leq \frac{\text{const}}{r}$.
 Suppose $K - N = A + \partial B$. Obviously $K - N(f\varphi) \geq \text{vol}(B(p, r/2) \cap K)$
 $\geq \text{const} \cdot r^k$ by theorem 2.0.1
 Also $K - N(f\varphi) = A(f\varphi) + B(df \wedge \varphi) \leq \mathbf{M}(A) + \frac{\text{const} \cdot \mathbf{M}(B)}{r} \leq \frac{\text{const} \cdot (\mathbf{M}(A) + \mathbf{M}(B))}{r}$.
 So taking the infimum we get

$$\text{const} \cdot r^k \leq \frac{\mathcal{F}(K - N)}{r}$$

which is the statement of the Corollary. Q.E.D.
 From that we get an immediate corollary

Corollary 2.0.2 : *If a sequence of submanifolds L_i in Φ converges to a current L , then it converges to the support of L in Gromov-Hausdorff topology .*

We now come to the main result of this section. We wish to strengthen Theorem 2.0.1 by an analogous result for volume of balls of radius r in the induced metric on calibrated submanifolds (which are smaller then the balls we considered before). We have the following

Theorem 2.0.2 : *Let M, φ, p, L, K and $B^K(r)$ as in Theorem 2.0.1 and let d_L be a distance function to p on L in the induced metric on L . Then for $r \leq \min(\text{inrad}(M), R(K))$ we have :*

$$\text{vol}(x \in L | d_L(x) \leq r) \geq \text{vol}(B^K(r))$$

and $R(K) = \pi/\sqrt{K}$ for K positive.

Corollary 2.0.3 : *Let M, φ as before. Then there is an a priori bound on a diameter of calibrated submanifolds in a given homology class η .*

Proof of the Corollary: Choose some r satisfying conditions of theorem 2.0.2. Let L be some calibrated submanifold in a homology class η . Let Γ be a maximal covering of L by disjoint balls of radius r . Since by theorem 2.0.2 each such ball has a volume at least ϵ and the volume of L is $v = [\varphi](\eta)$, then such covering exists and the number of elements in Γ is at most $N = \frac{v}{\epsilon}$. Now every point in L is contained in one of the balls of radius $2r$ with the same centers as balls in Γ .

So it is easy to deduce that the diameter of L is at most $4rN$. Q.E.D.

Proof of Theorem 2.0.2: We wish to use the same argument as in the proof of Theorem 2.0.1 for the distance function d_L . The problem is that d_L is not a smooth function in the r -neighbourhood of p . But we can still smoothen it using the following technical Lemma:

Lemma 2.0.2 : *Let L be a submanifold, $p \in L$ and d_L as before. We can pick $\rho > 0$ and a (C^∞) function $0 \leq \nu \leq 1$ on $[0, \infty)$ which is 0 on $(0, \rho]$, 1 on $[2\rho, \infty)$ and nondecreasing s.t. for any positive ϵ there is a function λ_ϵ on L which satisfies :*

- 1) λ_ϵ is C^∞ outside of p
- 2) $d_L \leq \lambda_\epsilon \leq d_L(1 + \epsilon)$
- 3) $|\nabla \lambda_\epsilon| \leq 1 + \nu(d_L)\epsilon$

Proof: Pick a positive $\rho \ll \text{in}jrad(L)$. Choose a function κ on M s.t. $\kappa = 1$ on $B(p, 2\rho)$ and $\kappa = 0$ outside of $B(p, 3\rho)$. Choose a nonnegative radially symmetric function σ on \mathbb{R}^k with support in the unit ball which integrates to 1 and let $\sigma_n(x) = n^k \cdot \sigma(nx)$. Then σ_n also integrates to 1. Choose a nonnegative function $\eta \leq 1$, $\eta = 0$ on $B(p, \frac{5\rho}{4})$ and $\eta = 1$ outside of $B(p, \frac{3\rho}{2})$.

Define now $\mu^n : L \mapsto \mathbb{R}$,

$$\mu^n(q) = \int_{T_q L} d_L(\exp(\theta)) \sigma^n(\theta) d\theta$$

Here $T_q L$ is the tangent bundle to q at L . Since σ^n was radially symmetric function and $T_q L$ has a metric, the expression $\sigma^n(\theta)$ is well defined and also integration takes part only on a ball of radius $\frac{1}{n} \subset T_q L$. Also it is clear that

$$\mu^n = d_L + o\left(\frac{1}{n}\right)$$

The point is that for large n , μ^n is a smooth function on L . Indeed let us denote by $J(a, b)$ the Jacobian of exponential map from a that hits b for a, b points in L that are close enough. Then $J(a, b)$ is a smooth function on (a, b) and we can rewrite

$$\mu^n(q) = \int_L J(q, b)^{-1} \cdot d_L(b) \cdot \sigma(\exp_b^{-1}(q)) db$$

and it is clear from this definition that μ^n is a smooth function of q for n large enough.

Also one can easily prove that $|\mu^n(q_1) - \mu^n(q_2)| \leq d(q_1, q_2) \cdot (1 + o(\frac{1}{n}))$, hence

$$|\nabla \mu^n| \leq 1 + o\left(\frac{1}{n}\right)$$

Now pick $\epsilon > 0$. Define $\lambda_\epsilon^n = (1 + \eta\epsilon)(\kappa \cdot d_L + (1 - \kappa) \cdot \mu^n)$.

Then $\lambda_\epsilon^n = d_L$ on $B(p, \frac{3\rho}{2})$ and it is smooth outside of p .

One can also directly verify that we can choose a constant C s.t. for sufficiently large n , the function $\lambda_\epsilon = \lambda_\epsilon^n$ satisfies properties 2) and 3) as desired. Q.E.D.

Now we can prove **Theorem 2.0.2:** We will use the fact that the function $\alpha(t)$, defined in the proof of Theorem 2.0.1, is an increasing function of t for $0 \leq t \leq \frac{\pi}{\sqrt{K}}$ for K positive and for $0 \leq t \leq R(K)$ for K negative.

Pick ρ as in Lemma 2.0.2. Let $\epsilon > 0$. We will follow the lines of proof of Theorem 2.0.1 for the function λ_ϵ instead of the distance function. We denote by

$$f(t) = \text{vol}(\lambda_\epsilon^{-1}([0, t])) , \quad S_t = \lambda_\epsilon^{-1}(t)$$

Then conditions on λ_ϵ and the co-area formula imply that for a regular value t we have $f'(t) \geq \frac{\text{vol}(S_t)}{1+\epsilon\nu(t)}$.

Also we can consider $A_t = ((a, \theta) | a \in S_t, 0 \leq \theta \leq d_p(a))$ (here d_p is the distance to p in the ambient manifold). We have $\xi : A_t \mapsto M$, $\xi(a, \theta) = \exp_M(\frac{\theta \cdot \exp^{-1}(a)}{d_p(a)})$. As before we will have $f(t) \leq \text{vol}(\xi(A_t))$ and $Jac(d\xi) \leq (F_{d_p(a)}(\theta))^{k-1}$ (see (3), we have the same notations as in Theorem 2.0.1). The estimate for Jacobian is true for the following reason: Let v_1, \dots, v_{k-1} be an o.n. basis to S_t at a . Then only the normal component of $d\xi(v_i)$ to the geodesic contributes to $Jac(d\xi)$. The normal component can be estimated by Lemma 2.0.1.

So we will have $\text{vol}(\xi(A_t)) \leq \int_{S_t} \alpha(d_p(a)) da \leq \text{vol}(S_t) \cdot \alpha(t)$ (here we used the fact that α is an increasing function and $d_p(a) \leq d_L(a) \leq \lambda_\epsilon(a) = t$). Combining all this we get

$$(\ln f)'(t) \geq \frac{(\ln \bar{f})'(t)}{1 + \epsilon\nu(t)} = [(\frac{\ln \bar{f}}{1 + \epsilon\nu})' + \epsilon\nu'/(1 + \epsilon\nu)^2 \cdot \ln(\bar{f})](t)$$

Now $\nu(t) = 0$ for $t \leq \rho$ and $\nu'(t) = 0$ for $t \geq 2\rho$ and $\ln(\bar{f}) \geq -C$ for $2\rho \geq t \geq \rho$. So

$$(\ln f)' \geq (\frac{\ln \bar{f}}{1 + \epsilon\nu})' - \epsilon C'$$

i.e. $(\ln f + \epsilon C't)' \geq (\frac{\ln \bar{f}}{1 + \epsilon\nu})'$

and for θ small we have $\ln(f(\theta) + \epsilon C'\theta) \geq \ln(\bar{f}(\theta)) = \frac{\ln \bar{f}(\theta)}{1 + \epsilon\nu(\theta)}$.

So $\ln(f + \epsilon C't) \geq \ln \bar{f}/(1 + \epsilon\nu)$.

Here ϵ was arbitrary and we are done. Q.E.D.

3 Special Lagrangian geometry on a Calabi-Yau manifold

3.1 Basic properties and examples

Let M^{2n} be a Calabi-Yau manifold, φ a holomorphic volume form and ω a Kahler 2-form. If ω is a Calabi-Yau form then $\text{Re}(\varphi)$ is a calibration (see [14]) and calibrated submanifold L can be characterized by alternative conditions : $\omega|_L = 0$ and $\text{Im}(\varphi)|_L = 0$. For arbitrary Kahler form ω we can define special Lagrangian (SLag) submanifolds by those 2 conditions. The form φ has length f with respect to the metric ω (here f is a positive function on M). We can conformally change the metric so that the form φ will have length $\sqrt{2}^n$ with respect to the new metric g' . Then SLag submanifolds will be Calibrated by $\text{Re}\varphi$ with respect to g' .

Lemma 3.1.1 *Let L^n be a compact connected n -dimensional manifold. Then the moduli-space of SLag embeddings of L into M is a smooth manifold of dimension $b_1(L)$.*

Proof: The proof is a slight modification of McLean's proof for a Calabi-Yau metric (see [14]).

Let $i : L \mapsto M$ be a (smooth) SLAG embedding of L into M . Locally the moduli-space Γ of $C^{2,\alpha}$ -embeddings of L into M (modulo the diffeomorphisms of L) can be identified with the $C^{2,\alpha}$ sections of the normal bundle of $i(L)$ to M via the exponential map. Also the normal bundle is naturally isomorphic to the cotangent bundle of L via the map $v \mapsto i_v^* \omega$. Hence the tangent bundle to Γ can be identified with $C^{2,\alpha}$ 1-forms on L . Let V_k be the vector space of exact $C^{1,\alpha}$ k -forms on L and let $V = V_2 \oplus V_n$. There is locally a map $\sigma : \Gamma \mapsto V$, given at an embedding $j(L) \in \Phi$ by $(j^*(\omega), j^*(Im\varphi))$. The moduli-space Φ of SLAG embeddings is just the zero set of σ . Now the differential of σ at $i(L)$ in the direction of α (there α is a $C^{2,\alpha}$ 1-form on L as above) is

$$(d\alpha, d(f * \alpha))$$

there f is the length of φ in the metric defined by ω . For ω a Calabi-Yau metric f is constant. We claim that the differential is surjective and the tangent space to Φ is naturally isomorphic to the first cohomology $H^1(L, \mathbb{R})$. To prove this consider first an operator P from the space of $C^{3,\alpha}$ functions on L to $C^{1,\alpha}$ n -forms on L , $P(h) = d(f * dh)$. We claim that P is surjective onto the space of exact n -forms and the kernel of P is the space of constant functions on L . Since f is non-vanishing P is elliptic. So to prove the surjectivity it is enough to show that the co-kernel of P consists of constant multiples of the volume form on L . Let μ be in the co-kernel of P . Let $h = *\mu$. One easily computes that

$$\int_L Ph \cdot \mu = \pm \int_L f |d^*(\mu)|^2$$

So $d^*(\mu) = 0$, hence μ is a constant multiple of the volume form on M . Let now h be in the kernel of P . Then arguing as before we get that $\mu = *h$ is a constant multiple of the volume form on M , i.e. h is a constant.

Now we can prove the lemma. First we prove that $d\sigma$ is surjective. Let α be an exact 2-form on L , and β be an exact n -form on L . We need to find a 1-form γ on L s.t.

$$d\gamma = \alpha, \quad d(f * \gamma) = \beta$$

Since α is exact there is a 1-form γ' s.t. $d\gamma' = \alpha$. We are looking for γ of the form $\gamma = \gamma' + dh$ for a function h . Since the operator P was surjective onto the space of exact n -forms on L , we get that such h exists, so $d\sigma$ is surjective, hence Φ is smooth. Next we prove that $\dim(\Phi) = b_1(L)$. Let $W = \ker(d\sigma)$. W is the tangent space to Φ at $i(L)$. Since W is represented by closed 1-forms on L , there is a natural map $\xi : W \mapsto H^1(L, \mathbb{R})$. We claim that this map is an isomorphism. Indeed let $a \in H^1(L, \mathbb{R})$ and let γ' be a closed 1-form on L representing the class a . From the properties of operator P it is clear that there is a unique exact 1-form $\gamma'' = dh$ s.t. $\gamma = \gamma' + \gamma''$ is in the kernel of σ . Hence ξ is an isomorphism Q.E.D.

Remark: A more general setup of deformations of SLAG submanifolds in a symplectic manifold was considered by S. Salur in [20].

In all subsequent discussions the moduli-space will be connected (i.e. we take a connected component of the moduli-space of SLag embeddings of a given manifold L).

We can also define Φ' - a moduli-space as special Lagrangian embeddings of a given manifold L into M over Diff' (diffeomorphisms of L which induce identity map on the homology of L). Then Φ' is a covering space of Φ . Now any element α in the first homology of L induces a 1-form h^α on Φ' in the following way : Let $\xi \in \Phi'$ and let L_ξ be it's support in M . If v is a tangent vector to Φ' at ξ then we can view v as a closed 1-form on L_ξ . From definition of Φ' it is clear that the element α induces a well defined element in $H_1(L_\xi)$, which we will also call α . So we define $h^\alpha(v) = [v](\alpha)$. Hitchin effectively proved in [8] that h^α is a closed form on Φ' (his notations are somewhat different from ours). Thus if we pick $\alpha_1 \dots \alpha_k$ a basis for the first homology of L then we have a frame of closed forms $h^1 \dots h^k$ and correspondingly a dual frame of commuting vector fields $h_1 \dots h_k$ on Φ' . Hence any compact connected component Γ of Φ must be a torus. Indeed the flow by commuting vector fields h_i induces a transitive \mathbb{R}^k action on Γ with stabilizer being a discrete subgroup A , hence Γ is diffeomorphic to \mathbb{R}^k/A - a k-torus.

Next we investigate finite group actions on Calabi-Yau manifolds. Suppose that a group G acts by structure preserving diffeomorphisms on M . We have the following

Lemma 3.1.2 : *Suppose a SLag submanifold L is invariant under the G action and G acts trivially on the first cohomology of L . Then G leaves invariant every element in the moduli-space Φ through L .*

Moreover, suppose that $g \in G$ and $x \in M - L$ in an isolated fixed point of g . Then x cannot be contained in any element of Φ

Proof : Since G is structure preserving, it sends SLag submanifolds to SLag submanifolds. Since it leaves L invariant, it preserves Φ (which is a connected component of L in the moduli-space of SLag submanifolds). From the identification of the tangent space of Φ at L with $H^1(L, \mathbb{R})$ and the fact that G acts trivially on $H^1(L, \mathbb{R})$ we deduce that G acts trivially on the tangent space to Φ at L . Hence G acts trivially on Φ , i.e. it leaves each element of Φ invariant.

To prove the second statement, consider a set S of those elements in Φ which contain x . Obviously S is closed and doesn't contain L . We prove that S is open and then it will be empty.

Let $L' \in S$. Any element L'' close to L' can be viewed uniquely as an image $\exp(v)$, where v is a normal vector field to L' . Suppose $v(x) \neq 0$. Since L'' is g -invariant then $\exp(g_*v(x))$ is also in L'' , there g_* is a differential of g at x . Since L' is g -invariant then g_* preserves the tangent space to L' at x , hence it preserves the normal space. Also since x is isolated then g_* has no nonzero invariant vectors. Hence $v(x) \neq g_*v(x)$ in the normal bundle.

Since exponential map is a diffeomorphism from a small neighbourhood of the normal bundle of L' to M we see that $\exp(g_*v(x))$ is not in L'' - a contradiction. So $v(x) = 0$ i.e. $L'' \in S$ Q.E.D.

As for examples of special Lagrangian submanifolds, many come from the following setup : Let M be a Calabi-Yau manifold and σ an antiholomorphic involution. Suppose σ reverses ω . Then the fixed-point set of σ is a special Lagrangian submanifold. For a Calabi-Yau metric ω the condition σ reverses ω is equivalent to σ reversing the cohomology class $[\omega]$, which often can be easily verified. Indeed suppose σ reverses $[\omega]$. Then $-\sigma^*(\omega)$ is easily seen to define a Kahler form, which lies in the same cohomology class as ω and the metric it induces is obviously equal to $\sigma^*(g)$, i.e. it is Ricci-flat. Hence by Yau's fundamental result (see [25]) we have $-\sigma^*(\omega) = \omega$.

We wish to discuss 2 collections of such examples. In both cases M is a projective manifold defined as a zero set of a collection of real polynomials. Then the conjugation of the projective space induces an anti-holomorphic involution which reverses the Fubini-Study Kahler form, hence it also reverses the Calabi-Yau form in the same cohomology class. The fixed point set is a submanifold of a real projective space .

Our first example will be a complete intersection of hypersurfaces of degree 4 and 2 in $\mathbb{C}P^5$.

First we note that a 2-torus can be represented as surface of degree 4 in \mathbb{R}^3 . Indeed a torus can be viewed as a circle bundle over a circle $((x, y, 0) | x^2 + y^2 = 1)$ in \mathbb{R}^3 , there a fiber over a point $a = (x, y, 0)$ is a circle of radius $\frac{1}{2}$ centered at a and passing in a plane through $a, (0, 0, 1)$ and the origin . If (x, y, z) is a point on our torus then it's distance to a point $(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0)$ is $\frac{1}{2}$.

If we compute we get $1 + x^2 + y^2 + z^2 - 2\sqrt{x^2 + y^2} = \frac{1}{4}$, i.e. $p(x, y, z) = (\frac{3}{4} + x^2 + y^2 + z^2)^2 - 4(x^2 + y^2) = 0$.

So in inhomogeneous coordinates $x_1 \dots x_5$ on $\mathbb{R}P^5$ the zero locus of 2 polynomials $p(x_1, x_2, x_3)$ and $q(x_4, x_5)$ (there $q(x, y) = x^2 + y^2 - 1$) is a 3-torus. If we consider the corresponding homogeneous polynomials on $\mathbb{R}P^5$, then it is easy to see that there is no solution for $x_6 = 0$. So the zero locus of those polynomials in $\mathbb{R}P^5$ is a 3-torus. If we perturb them slightly so that the corresponding complex 3-fold in $\mathbb{C}P^5$ will be smooth then we obtain the desired example.

Other examples are quintics with real coefficients in $\mathbb{C}P^4$. In that case real quintics would be special Lagrangian submanifolds. R.Bryant constructed in [3] a real quintic, which is a 3-torus .

We will construct, using Viro's technique in real algebraic geometry (see [23]), real quintics L_k which are diffeomorphic to projective space $\mathbb{R}P^3$ with k 1-handles attached for $k = 0, 1, 2, 3$. If $k = 3$ then $b_1(L_3) = 3$ and the cup product in the first cohomology of L_3 is 0.

The construction goes as follows: First in inhomogeneous coordinates x_1, \dots, x_4 on $\mathbb{R}P^4$ we consider a polynomial $p \cdot q$, there $p(x_1, \dots, x_4) = x_1^2 + \dots + x_4^2 - 1$ and $q = x_4$. In $\mathbb{R}P^4$ the zero locus of the polynomial will be $\mathbb{R}P^3 \cup S^3$, there $\mathbb{R}P^3$ is a zero locus of q , S^3 is a zero locus of p and $\mathbb{R}P^3$ intersects S^3 along a 2-sphere $S^2 \subset \mathbb{R}^3 = (x_4 = 0)$.

Now we consider $f = pq - \epsilon h$, there h is some polynomial of degree up to 5 and $\epsilon > 0$ is small enough. The Hessian of pq on S^2 is nondegenerate along the normal bundle to S^2 and vanishes along the axes of the normal bundle (the axes are a normal bundle of S^3 to S^2 and of $\mathbb{R}P^3$ to S^2). If we look on those axes locally as coordinate axes then $pq = xy$ in those coordinates.

Suppose first h is non-zero along S^2 . We can assume that $h > 0$ on S^2 . The zero locus of $f = pq - \epsilon h$ will live in 2 quadrants in which $xy > 0$. Thus in zero locus of f , a part A_1 of $\mathbb{R}P^3$ outside S^2 will "connect" with one hemisphere S' of S^3 , and the part A_2 inside S^2 will connect with the other hemisphere S'' and the zero locus of f is a disjoint union of $\mathbb{R}P^3$ and S^3 .

Suppose the zero set of h intersects our S^2 transversally along k circles such that no circle will lie in the interior of the other. We can assume w.l.o.g. that on the exterior V of those circles h is positive. Then along V , A_1 connects to S' and A_2 connects to S'' as before. Along the interior of every circle, A_1 connects to S'' and A_2 to S' . So near interior of these circles we get 1-handles connecting $\mathbb{R}P^3$ with S^3 . So the zero locus of f will be $\mathbb{R}P^3$ and S^3 connected by k 1-handles, i.e. it will be an $\mathbb{R}P^3$ with $k - 1$ 1-handles attached.

It is not hard to find examples of such h for small values of k . For instance for $k = 4$ (i.e. for L_3) we can take

$$h = ((x_1 - 1/3)^2 + (x_2 - 1/3)^2 - 1/16)((x_1 + 1/3)^2 + (x_2 + 1/3)^2 - 1/16)$$

and the zero locus of h intersects $S^2 \subset \mathbb{R}^3$ in 4 circles.

3.2 Non-compactness of the moduli-space

In this section we will consider connections between the moduli-space of SLAG submanifolds and global geometry of the ambient Calabi-Yau manifold M .

Let Φ be the moduli-space of SLAG submanifolds. We have a fiber bundle F over Φ , $F \subset M \times \Phi$, $F = \{(a, L) | a \in M, L \in \Phi \text{ s.t. } a \in L\}$.

We have a natural projection map $pr : F \rightarrow \Phi$, whose fiber is the support of the element in Φ and the evaluation map $ev : F \rightarrow M$, $ev(a, L) = a$.

Also the tangent space to a point $(a, L) \in F$ naturally splits as $T_a L \oplus T'$, there $T_a L$ is the tangent space to L at a (a tangent space to the fiber) and $T' = ((v(a), v) | v \text{ is a variation v. field to the moduli-space and } a(v) \text{ is the value of } v \text{ at } a)$.

Let L be a compact k -dimensional oriented manifold with $b_1(L) = k$. We say that L satisfies condition \star if for $\alpha_1 \dots \alpha_k$ a basis for $H^1(L)$ we have $\alpha_1 \cup \dots \cup \alpha_k \neq 0$. This holds e.g. if L is a torus. On the other hand the real quintic with $b_1 = 3$ that we constructed in section 3.1 doesn't satisfy condition \star .

Theorem 3.2.1 : *Let L be a special Lagrangian submanifold with $b_1(L) = k$. Suppose L satisfies \star . Suppose some connected component of Φ' is compact. Then the Betti numbers of M satisfy : $b_i(M) \leq b_i(L \times T^k)$ (here T^k is a k -torus).*

Suppose we have G, g, x satisfying conditions of Lemma 3.1.1. Then Φ itself is not compact.

Proof of Theorem 3.2.1: Suppose L satisfies \star . First prove that Φ is orientable (in fact it has a natural volume element σ). Let $L' \in \Phi$ and $v_1 \dots v_k$ be elements of the tangent space to Φ at L' . So $v_1 \dots v_k$ are closed 1-forms on L' and we define:

$$\sigma(v_1 \dots v_k) = [v_1] \cup \dots \cup [v_k](L')$$

Suppose that Φ is compact, or a connected component Γ of Φ' is compact. In each case we have an evaluation map as before. We will prove that in both cases it has a positive degree. We will give a proof for Φ , the proof for Γ is analogous.

First Φ has a natural volume element σ described above. So the $2k$ -form $\alpha = pr^*(\sigma) \wedge ev^*(Re\varphi)$ is the volume form on F .

Let $L_\phi \in \Phi$ and $\alpha_1 \dots \alpha_k$ be a basis for $H^1(L_\phi)$ s.t. $\alpha_1 \cup \dots \cup \alpha_k[L_\phi] = 1$. Then we can consider corresponding vector fields $v_1 \dots v_k$ along L_ϕ , which form a frame for the bundle T' (described in the beginning of this section) restricted to L_ϕ . So $[i_{v_j}\omega] = \alpha_j$ and $pr^*(\sigma)(v_1, \dots, v_k) = 1$.

Let now η be a Riemannian volume form on M . Then we have

$$deg(ev) = \int_F ev^*(\eta)/vol(M)$$

Since F is a fiber bundle we can use integration over the fiber formula to compute:

$$\int_F ev^*(\eta) = \int_\Phi \left(\int_{L_\phi} i_{v_1} \dots i_{v_k} ev^*(\eta) \right) d\phi$$

(of course we choose $\alpha_1 \dots \alpha_k$ for each L_ϕ).

Also $i_{v_1} \dots i_{v_k} ev^*(\eta)$ is easily seen to be equal to $i_{v_1}\omega \wedge \dots \wedge i_{v_k}\omega$ (all restricted to the fiber L_ϕ).

So $\int_{L_\phi} i_{v_1} \dots i_{v_k} ev^*(\eta) = \alpha_1 \cup \dots \cup \alpha_k(L_\phi) = 1$.

So $deg(ev) = \int_\Phi 1/vol(M) = vol(\Phi)/vol(M) > 0$.

Now let Γ be compact. Let F' be a corresponding fiber bundle over Γ . First we claim that $b_i(F') \geq b_i(M)$. Suppose this is not true for some $0 < i < k$. Then we have $\beta \in H^i(M)$ s.t. $ev^*(\beta) = 0 \in H^i(F')$. By Poincare duality we can find $\alpha \in H^{k-i}(M)$ s.t. $\alpha \cup \beta$ is the generator of $H^k(M)$. Since $deg(ev) \neq 0$ we have $0 \neq ev^*(\alpha \cup \beta) = ev^*(\alpha) \cup ev^*(\beta) = 0$ - a contradiction.

Next we prove that $b_i(F') \leq b_i(L \times T^k)$. We know from Section 3.1 that Γ is a k -torus. Also if we fix a basis for each cohomology $H^i(L, \mathbb{R})$ then each fiber of the fibration F' over Γ has a canonical basis for cohomology (because Γ is a space of SLAG embeddings of a manifold L modulo diffeomorphisms which induce identity map in homology).

Let U_α be a good cover of Γ . Then $pr^{-1}(U_\alpha)$ is a cover of F' . Consider a corresponding double complex (see [2], p.96) $K = K^{p,q} = C^p(pr^{-1}(U_\alpha)_q)$ (here $pr^{-1}(U_\alpha)_q$ is a collections of all q intersections of elements in $pr^{-1}(U_\alpha)$).

Then this complex computes the DeRham cohomology of F' . Also the cohomology of this complex is computed by the corresponding Leray spectral sequence (see [2], p.165) $E_r^m = \oplus E_r^{p,q}$ and the second term is given by $E_2^{p,q} = H_\delta^p(H_d^q(K))$, there H_d^q is a sheaf over Γ given by $H_d^q(U) = H^q(pr^{-1}(U))$. Since

each fiber has a canonical basis for cohomology we see that H_d^q is a constant sheaf $H^q(L)$ over Γ and hence $E_2^{p,q} = H^q(L) \otimes H^p(\Gamma)$, so $\dim(E_2^m) = b_m(Y \times T^k)$. So we have $b_i(M) \leq b_i(F') = \dim(E_\infty^m) \leq \dim(E_2^m) = b_i(L \times T^k)$.

Finally suppose G, g, x satisfy conditions of Lemma 3.1.1. Suppose Φ is compact. Then the degree of the evaluation map is positive, hence it is surjective. But x is not in the image of the evaluation map- a contradiction. Q.E.D.

3.3 Special Lagrangian fibration on a Borcea-Voisin threefold

In this section we will use results of 2 previous sections to investigate one example of a Calabi-Yau manifold in detail. We study the Borcea-Voisin threefold M . It will turn out that for a suitable choice of a Kahler metric on M we can prove that M fibers with a generic fiber being a Special Lagrangian torus. Moreover we construct a mirror fibration to our Special Lagrangian fibration.

The property, which we will utilize in studying M , is the fact that M is a union of neighbourhoods, each of those is biholomorphic to a product of an elliptic curve with an open neighbourhood on a K^3 surface. This will enable us to define a certain Kahler metric on M such that in any such neighbourhood SLag submanifolds in our moduli-space will look like $S^1 \times T$, where S^1 is a circle and T is a pseudoholomorphic 2-torus. This will enable us to use Gromov's compactness theorem to study the compactification $\bar{\Phi}$ of the moduli-space Φ of SLag tori. A crucial point will be the fact that the boundary $\bar{\Phi} - \Phi$ has dimension 1 (i.e. co-dimension 2). So the total space F (see section 3.2) compactifies to a pseudo-cycle (see [21]). Moreover both F and M are orientable. So we can define the degree of the evaluation map $\deg(ev)$ as follows:

Let $x \in M$ be a regular point (i.e the evaluation map is transversal to x and x is not contained in the total space of the boundary of the moduli-space). Then $\deg(ev)$ is a sum over the preimages $ev^{-1}(x)$ of their signs. By usual transversality this is well-defined and in our example we will have $\deg(ev) = 1$, i.e. the moduli-space fills the whole manifold M .

We will adopt the definition of M from [13], so we define M to be the resolution of singularities of a 6-torus $T^6 = T^2 \times T^2 \times T^2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, there the generators of the \mathbb{Z}_2 actions are :

$$\alpha : z_1 \rightarrow -z_1 + \frac{1}{2}, z_2 \rightarrow -z_2 + \frac{1}{2}, z_3 \rightarrow z_3$$

and

$$\beta : z_1 \rightarrow -z_1, z_2 \rightarrow z_2, z_3 \rightarrow -z_3$$

The fixed locus of α is 16 2-tori :

$$A \times A \times T^2$$

and the one of β is 16 2-tori:

$$B \times T^2 \times B.$$

Here $A \subset T^2$ is a set $\{\frac{2+2i+1+i}{4}\}$ and $B \subset T^2$ is a set $\{0, \frac{1}{2}, \frac{i}{2}, \frac{i+1}{2}\}$.

The fixed loci of α and β do not intersect. Also $\alpha \circ \beta$ has no fixed points.

Consider a fixed torus T^2 (say of α). Near T^2 the quotient looks like $V = (U/\pm 1) \times T^2$, there U is a ball of radius r around a fixed point in T^4 (we can also view it as a neighbourhood of the origin in \mathbb{C}^2).

The resolution of singularity $U/\pm 1$ is an r -neighbourhood \overline{U} of the zero set in the total space of $\gamma^{\otimes 2}$, there γ is the universal line bundle over $\mathbb{C}P^1$ (thus the singular point will be replaced by $\mathbb{C}P^1$). The resolution has a 1-parameter family of HyperKahler metrics ω_t (the Eguchi-Hanson metrics) (see [10], p.304). Their Kahler potentials f_t are given by

$$f_t(u) = \sqrt{u^2 + t^2} + t^2 \log u - t^2 \log(\sqrt{u^2 + t^2} + t^2)$$

here $u = |z_1|^2 + |z_2|^2$. So we replace $V = (U/\pm 1) \times T^2$ by $\overline{V} = \overline{U} \times T^2$. Now the Eguchi-Hanson Kahler form ω_t can be glued to a Euclidean metric inside of $U/\pm 1$ by gluing their Kahler potentials for t small enough and we obtain a Kahler form ω' on \overline{U} . We consider the corresponding product metric in \overline{V} . Doing this for every fixed 2-torus (both of α and of β) we get a Kahler metric on M , which is Euclidean outside neighbourhoods described above.

If we take a family of 3-tori $T_{a,b,c} \subset M$,

$$T_{a,b,c} = ((z_1, z_2, z_3) | \operatorname{Re} z_1 = a, \operatorname{Re} z_2 = b, \operatorname{Re} z_3 = c)$$

which don't intersect neighbourhoods of fixed components, then they will be SLAG tori in M (according to the definition in the beginning of section 3.1). We would like to see what happens to this family then it's elements intersect some neighbourhood \overline{V} described above.

We wish to point out that P. Lu considered SLAG submanifolds for a Calabi-Yau metric on M in [13]. He was able to produce a big open set of those submanifolds. We are using a different metric and this will allow us to characterize the compactified moduli-space and to prove that M fibers over it.

We return to the question of characterizing those elements which intersect a 'bad' neighbourhood. If this is a neighbourhood of a fixed component of α , we will consider the following setup :

Consider a \mathbb{Z}_2 action on T^4 with a generator

$$\alpha' : z_1 \rightarrow -z_1 - \frac{1}{2}, z_2 \rightarrow -z_2 - \frac{1}{2}$$

Then this action has 16 fixed points and the resolution of singularities gives a K^3 surface. Also our original manifold can be viewed as

$$(T^4/\alpha' \times T^2)/\beta$$

For each fixed point of α' we introduce a neighbourhood U as before. We will have to consider a bigger neighbourhood $X = ((z_1 + ia, z_2 + ib) | (z_1, z_2) \in U, a, b \in \mathbb{R}/\mathbb{Z})$ in T^4 and a corresponding neighbourhood in the quotient, which we call X' . We will consider a resolution of singularities \overline{X} (now we have 4 singular points in X') and a corresponding domain $\overline{W} = \overline{X} \times T^2$ in M (since $\beta(\overline{W}) \cap \overline{W} = \emptyset$, we can view \overline{W} as a domain in M).

Consider a canonical $(2,0)$ form $\eta = \text{Re}\eta + i\text{Im}\eta$ on $X \subset T^4$ (The collection $w, \text{Re}\eta, \text{Im}\eta$ is the standard HyperKahler package on T^4). Then η lifts to a holomorphic $(2,0)$ form on \overline{X} and we have $i \cdot \eta \wedge dz_3 = \varphi$ - a holomorphic $(3,0)$ form on M .

Since the Calabi-Yau structure on \overline{W} is a product structure, then in \overline{W} SLAG submanifolds, which come from a connected component of a family $T_{a,b,c}$, will look like : $L \times T_c$

there $T_c = \{z | \text{Re}z = c\} \subset T^2$ and L is a SLAG submanifold of \overline{X} w.r. to ω', η . Indeed submanifolds described above form a 3-dimensional moduli-space, which is contained in the original moduli-space, hence it coincides with the connected component of the original moduli-space.

The package $\omega', \text{Re}\eta, \text{Im}\eta$ is not a HyperKahler package on \overline{X} . We can however normalize ω' to some ω'' (by multiplying it by a positive function) s.t. in the metric defined by ω'' the form η will have length $\sqrt{2}$. In this case $\omega'', \text{Re}\eta, \text{Im}\eta$ will be a (non-integrable) HyperKahler package on \overline{X} .

A SLAG submanifold $L \subset \overline{X}$ is defined by conditions $\omega'|_L = 0, \text{Im}\eta|_L = 0$, which is of course equivalent to $\omega''|_L = 0, \text{Im}\eta|_L = 0$, hence L is a SLAG submanifold with respect to our HyperKahler package, which is equivalent to being pseudoholomorphic submanifold with respect to (non-integrable) almost complex structure defined by $\text{Re}\eta$.

Now elements in our moduli-space on M have trivial self-intersection. Since those elements, which live in \overline{W} look like $L \times T_a$ and L is pseudoholomorphic in \overline{X} , it is clear that they do not intersect. Also to characterize the moduli-space, it is obviously enough to study pseudoholomorphic tori in \overline{X} . Indeed the boundary of \overline{X} is fibered by pseudoholomorphic tori. Hence one easily deduces that an element in our moduli-space is either contained in \overline{W} or in it's complement.

We return now to our pseudoholomorphic tori living in \overline{X} . We can view those tori as living in K^3 as before. All of those tori will carry the same homology class h . We would like to study the boundary of our moduli-space, i.e. to understand what are possible limits of a sequence of such tori. A limit of any sequence T_i of such tori, by Gromov's compactness theorem (see [21] or [24]), is a cusp curve with at most 1 component being a torus and the rest are pseudoholomorphic spheres.

Suppose we have 1 component being a torus T . We want to prove that T is the only component, it is embedded and lives in the original moduli-space and T_i converge to T in the moduli-space.

We can represent T as a composition $\alpha \circ \rho : T^2 \mapsto K^3$, there $\rho : T^2 \mapsto T^2$ is a k -fold covering and $\alpha : T^2 \mapsto K^3$ is a simple curve (see [15], p.18).

Let $T' = \alpha(T^2)$. Since T doesn't intersect any of the T_i we have that $[T'] \cdot h = 0$. Also $h = k[T'] + \Sigma[S_i]$ for some pseudoholomorphic spheres S_i .

We have $[T'] \cdot \Sigma[S_i] \geq 0$ with equality iff there are no S_i (because the limiting curve is connected). Also since α is simple we get by theorem 7.3 in [17] that $[T'] \cdot [T'] \geq 0$ with equality iff T' is embedded. From all this we deduce that there are no rational components and T' is embedded.

Since T' is pseudoholomorphic, it is a SLAG torus, hence it admits a 2-dimensional deformation family of SLAG tori. Moreover this family fills some

neighbourhood of a point in T' . Indeed let α_1, α_2 be 2 generators of $H^1(T')$ and v_1, v_2 are corresponding deformation v.fields. Then $i_{v_1}\omega' \wedge i_{v_2}\omega'$ is nonzero in $H^2(T')$, so it doesn't vanish at some point $p \in T'$. Hence v_1 and v_2 are linearly independent at p , so the moduli space fills a neighbourhood U of p .

Also T_i converge to T in Gromov-Hausdorff topology, hence they intersect U for i large enough. So they intersect elements in the moduli-space through T' . Now these elements are pseudoholomorphic and carry a homology class h/k and $h \cdot h = 0$. So we deduce that T_i are in the moduli-space through T' and so $k = 1$ and $T = T'$ is in the original moduli-space. So the boundary of the moduli-space consists of unions of spheres.

Let Σ be a cusp curve on the boundary on the moduli-space. Then as before Σ doesn't intersect any of the smooth pseudoholomorphic tori. We want to prove that Σ doesn't intersect any over cusp-curve on the boundary of the moduli-space. To prove that we will have to understand the second homology $H_2(\bar{X}, \mathbb{Z})$ in detail. Suppose X (see p.14) is a neighbourhood of the torus $T = (Re(z_1) = 1/4, Re(z_2) = 1/4)$. Then in the quotient X' , T becomes a sphere S and one sees that S is a strong deformation retract of X' . The resolution of singularities \bar{X} is obtained from X' by replacing the singular points with the exceptional spheres S_i . Also there is a sphere S' in \bar{X} , which projects onto S . One can easily deduce from Mayer-Vietoris sequence that any element α in $H_2(\bar{X}, \mathbb{Z})$ can be represented as $\alpha = \lambda \cdot [S'] + \sum \lambda_i \cdot [S_i] + \text{torsion}$, there λ, λ_i are integers. Now $\int_{S'} Re(\eta) = 1/2 \cdot \int_T Re(\eta) = 1/2$ and $\int_{S_i} Re(\eta) = 0$. So $\int_\alpha Re\eta = \lambda \cdot 1/2 = \lambda/2$. So if $\int_\alpha Re\eta > 0$ then $\int_\alpha Re\eta \geq 1/2$.

Let now Σ be as before. Then Σ represents a homology class h and $\int_h Re(\eta) = 1$. Also the integral of $Re(\eta)$ on every component of Σ is at least $1/2$, so Σ has at most 2 components. Let Σ' be another cusp curve on the boundary of the moduli-space. Suppose Σ' intersects Σ . Since $h \cdot h = 0$ we see that Σ and Σ' must have a common component. Suppose Σ has a component P which is not in Σ' . Then $0 = [P] \cdot h = [P] \cdot [\Sigma'] > 0$ - a contradiction. So Σ and Σ' have same components, and since their total number (counted with multiplicity) is at most 2, then $\Sigma = \Sigma'$.

Finally we prove that the number of exceptional spheres is finite. As we have seen, there are 2 types of exceptional curves:

1) A curve with 2 components A_i and B_i . Then $0 = [A_i] \cdot h = [A_i] \cdot ([A_i] + [B_i])$. Now $[A_i] \cdot [B_i] > 0$, so $[A_i] \cdot [A_i] < 0$.

If A_j, B_j is another curve like that, then we have seen that A_i doesn't intersect it, so in particular $[A_i] \cdot [A_j] = 0$. So one easily sees that the numbers of such curves is at most $5 = b_2(\bar{X})$.

2) A curve with 1 component (possibly with multiplicities). Let this curve be $k \cdot P_i$, there P_i is a primitive rational curve and $k \cdot [P_i] = h$. To study those P_i we make following observations: There is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action on T^4 with generators

$$\gamma_1 : (z_1, z_2) \mapsto (z_1 + i/2, z_2)$$

$$\gamma_2 : (z_1, z_2) \mapsto (z_1, z_2 + i/2)$$

This action commutes with α' action, and hence induces an action on K^3 . It also preserves \bar{X} .

Next we find elements in K^3 , which do not have a full orbit under the action. A point (z_1, z_2) doesn't have a full orbit if it is preserved under one of $\gamma_1, \gamma_2, \gamma_1 \circ \gamma_2$. Now the fixed points are:

$Fix(\gamma_1) = ((z_1, z_2) : (z_1 + i/2, z_2) = (-z_1 + 1/2, -z_2 + 1/2))$. These are 2 points, disjoint from exceptional spheres. A similar analysis for γ_2 and $\gamma_1 \circ \gamma_2$ produces 2 points for each.

Now the actions of γ_i are structure preserving on \overline{X} , so they send SLAG tori to SLAG tori. Moreover they preserve an open set of tori $Re(z_i) = const$ in our moduli-space. So by Lemma 3.1.1 they leave elements of the moduli-space invariant. Hence they preserve the limiting curve P_i (because the convergence is in particular a Gromov-Hausdorff convergence).

For a limiting curve P_i , consider $\chi(P_i) = [P_i] \cdot [P_i] - c_1(K^3)([P_i]) + 2 = 2$. Then by theorem 7.3 of [17] we can count $\chi(P_i)$ by adding contributions of singular points (which are double points or branch points), and each singular point gives a positive contribution. So P_i has singular points and there are at most 2 of those.

Let x is a singular point. Then it's orbit under $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action consists of singular points. So it cannot have length 4, so x is one of 6 points D with orbit of length 2. So P_i contains at least 2 points of the set D .

If P_j is another curve of type 2, then $[P_i] \cdot [P_j] = 0$, so they don't intersect. Also P_j contains at least 2 points from the set D . So it is clear that the number of P_i is at most 3.

From all that we deduce that our moduli-space can be compactified to a pseudo-cycle. Also any point x outside of bad neighbourhoods has a unique preimage in the smooth part of the moduli-space, so we deduce that the degree of the evaluation map is 1, so the compactified moduli-space fills the whole manifold M . Also elements of the compactified moduli-space don't intersect, so M fibers with generic fiber being a SLAG torus. Also the fibration is smooth over the smooth part of the moduli-space. To prove that we need to prove that the differential of the evaluation map is an isomorphism. This is clearly true outside our 'bad' neighbourhoods. Inside a bad neighbourhood, it is enough to prove that variational vector fields to our pseudoholomorphic tori do not vanish. But this follows from a standard argument that each zero of such a vector field gives a positive contribution to the first Chern class of the normal bundle, which is trivial.

We want to point out that this example, then we can use HyperKahler trick and local product structure to study limiting SLAG submanifolds, is quite ad hoc and some new ideas are needed to study singular SLAG submanifolds in general.

Next we wish to construct a mirror, i.e. to compactify the dual fibration. Let

$\begin{array}{ccc} M & & M_0 \\ \downarrow & & \downarrow \\ \Phi & & \Phi \end{array}$ be a fibration over the compactified moduli-space and let $\downarrow \Phi$ be a restriction of this fibration over the (smooth) moduli-space, there $M_0 \subset M$ is an open subset. Let $a \in \Phi$ and L_a be a fiber. We have a vector space $V_a = H_1(L_a, \mathbb{R})$ and a lattice $\Lambda_a = H_1(L_a, \mathbb{Z})$ in it, and so we get a torus bundle V_a/Λ_a over Φ . By dualizing each V_a we get a dual bundle V_a^*/Λ_a^* . We will adopt the following

definition of a topological mirror from M. Gross's paper (see [4])

Definition 3.3.1 : Let $\begin{array}{c} M' \\ \downarrow \\ \Phi \end{array}$ be another fibration with M' smooth and a corresponding fibration $\begin{array}{c} M'_0 \\ \downarrow \\ \Phi \end{array}$ is a smooth torus fibration. Let V'_a/Λ'_a as before. We say that M' is a topological mirror to M if there is a fiberwise linear isomorphism $\rho : V_a^*/\Lambda_a^* \mapsto V'_a/\Lambda'_a$ over Φ .

Suppose now M is a symplectic manifold and $\begin{array}{c} M_0 \\ \downarrow \\ \Phi \end{array}$ is a Lagrangian fibration. Then Duistermaat's theory of action-angle coordinates (see [5]) implies that there is an action of the cotangent bundle $T^*\Phi$ on the fibers with a stabilizer lattice Λ_b . This of course induces a natural isomorphism $\xi : V_a/\Lambda_a \mapsto T^*\Phi/\Lambda_b$. There is also a dual isomorphism $\xi^* : T\Phi \mapsto V_a^*$ given explicitly by $\xi(v) = [i_v\omega]$, here ω is a symplectic structure and $v \in T\Phi$ is viewed as a normal vector field to an element of Φ . Also the natural symplectic structure on $T^*\Phi$ projects to a symplectic structure on $T^*\Phi/\Lambda_b$ and hence on V_a/Λ_a .

If our fibration is a Special Lagrangian fibration then one can get a symplectic structure on the dual bundle V_a^*/Λ_a^* as follows (this construction was done by Hitchin in [8] and in coordinate-free way by Gross in [5]):

We have a map $\alpha : V_a^* \mapsto T^*\Phi$ defined by periods of the closed form $Im\varphi$. Explicitly, let $u \in V_a^*$. We can view $u \in H^1(L, \mathbb{R})$. For $v \in T\Phi$ we define

$$\alpha(u)(v) = [i_v Im\varphi] \cup u([L]) = [i_v Im\varphi](PD(u)) \quad (4)$$

Here v is viewed as a normal vector field to L and $PD(u)$ is a Poincare dual to u . One shows that for u a section of Λ_a^* (the integral cohomology lattice), $\alpha(u)$ is a closed 1-form on Φ and thus α induces a symplectic structure on V_a^*/Λ_a^* . This motivates the following definition

Definition 3.3.2 Let $\begin{array}{c} M \\ \downarrow \\ \Phi \end{array}$ and $\begin{array}{c} M' \\ \downarrow \\ \Phi \end{array}$ be 2 Special Lagrangian fibrations. Then M' is a symplectic mirror to M if the corresponding isomorphism $\rho : V_a^*/\Lambda_a^* \mapsto V'_a/\Lambda'_a$ over Φ is a symplectomorphism.

To construct a topological mirror we make following observations: Let $F = \begin{array}{c} W_a/\Lambda_a \\ \downarrow \\ U \end{array}$ be some torus fibration. Let $a \in U$. Then we have a monodromy representation $\nu : \pi_1(U, a) \mapsto SL(W_a, \Lambda_a)$ (see [4]). Moreover if $F' = \begin{array}{c} W'_a/\Lambda'_a \\ \downarrow \\ U \end{array}$ is another fibration and $K : W_a \mapsto W'_a$ is an intertwining isomorphism for monodromy representations, then K induces a natural fiberwise isomorphism between F and F' .

So we can try to compactify a dual fibration by trying to find local isomorphisms between it and the original fibration. Let U be a neighbourhood in Φ

and $a \in U$. Let e_1, \dots, e_n be a basis for the lattice Λ_a and e^1, \dots, e^n be a dual basis for the lattice Λ_a^* . Let $K : W_a/\Lambda_a \mapsto W_a^*/\Lambda_a^*$ be some linear map, which is given in terms of our bases by a matrix K . Let $\alpha \in \pi_1(U, a)$ and $\nu(\alpha)$ be a monodromy map, which is given in terms of a basis (e_i) by a matrix A . Then a dual representation $\xi^*(\alpha)$ on W_a^* is given by a matrix $(A^T)^{-1}$ in a basis (e^i) (see [4]).

So we need $K \cdot A = (A^T)^{-1} \cdot K$ i.e. $K = A^T \cdot K \cdot A$.

Now if $n = 2$ there is a solution $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We return now to M . On a 6-torus T^6 we have a natural isomorphism between integral homology and cohomology of SLAG tori $T_{a,b,c}$. This isomorphism is invariant under $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action, hence it induces an isomorphism ρ between Λ_a and Λ_a^* outside of bad neighbourhoods. Take a point a on a boundary of a bad neighbourhood Y in Φ . Then because of the product structure of our fibration over Y we see that the monodromy matrices of V/Λ in Y look like

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is clear from the above that the monodromy representation in Y is isomorphic to a dual representation.

So we can construct a topological mirror M' as follows: Let \overline{W} be some 'bad' neighbourhood in M as before. Let $z_j = x_j + i \cdot y_j$ be coordinates on a 6-torus. Then a map $\mu : T^6 \mapsto T^6$, $\mu(x_1, y_1, x_2, y_2, x_3, y_3) = (x_1, -y_2, x_2, y_1, x_3, y_3)$ commutes with \mathbb{Z}_2^2 action. Also μ maps the boundary $\partial \overline{W}$ to itself. We take \overline{W} and glue it to $M - \overline{W}$ by μ and doing so for each 'bad' neighbourhood we obtain M' .

Now μ preserves SLAG tori on $\partial \overline{W}$, and thus M' naturally acquires a structure of a fiber bundle over $\overline{\Phi}$ with generic fiber being a torus. We claim that M' is a topological mirror of M .

Indeed we noted that outside of 'bad' neighbourhoods there is a natural isomorphism ρ between bundles V_a and V_a^* as before, and of course the bundle V'_a of M' is isomorphic to the bundle V_a outside of 'bad' neighbourhoods, so ρ can be viewed as an isomorphism between Λ_a^* and Λ'_a . We want to extend ρ inside of \overline{W} . First we need to check which isomorphism ρ induces on $\partial \overline{W}$ via the gluing map μ .

Let $L = T_{a,b,c}$ be a SLAG torus contained in $\partial \overline{W}$. Let $z_j = x_j + i \cdot y_j$ be coordinates on T^6 . Then dy_1, \dots, dy_3 is a basis for $H^1(L, \mathbb{Z}) = \Lambda_a^*$ and $\partial_{y_1}, \dots, \partial_{y_3}$ is a dual basis for $H_1(L, \mathbb{Z}) = \Lambda'_a$. Then

$$\rho : dy_1 \mapsto -\partial_{y_2}, \quad dy_2 \mapsto \partial_{y_1}, \quad dy_3 \mapsto \partial_{y_3}$$

As we saw, ρ is an intertwining operator between the monodromy representations on V_a^* and V'_a . Hence ρ extends to an isomorphism inside \overline{W} , and hence M' is a topological mirror of M .

So far we viewed M' just as a differential manifold (and one can easily show that M' is diffeomorphic to M). We will see that it has additional interesting structures.

Let ω', ω'' and $\eta = \text{Re}(\eta) + i \cdot \text{Im}(\eta)$ as before (see p.14). Then we easily see that near $\partial\overline{W}$ the gluing map μ is an isometry. Also $\mu^*(\omega') = \text{Im}\eta, \mu^*(\text{Im}\eta) = -\omega', \mu^*(\text{Re}\eta) = \text{Re}\eta$. Now $\text{Im}(\eta) + (dx_3 \wedge dy_3)$ is a symplectic form on \overline{W} . So we see that we can glue it to our symplectic form $\omega' + (dx_3 \wedge dy_3)$ outside of \overline{W} to get a symplectic form ω^* on M' . Moreover near $\partial\overline{W}$, μ intertwines between I and K - almost complex structures defined by ω' and $\text{Im}\eta$. Thus we can glue K inside \overline{W} to I outside of \overline{W} to get an almost complex structure I' on M' compatible with ω^* . We can glue a form $(\text{Re}\eta + i\text{Im}\eta) \wedge idz_3$ outside of \overline{W} to a form $(\text{Re}\eta - i\omega'') \wedge idz_3$ inside \overline{W} to get a trivialization φ' of a canonical bundle of I' .

A submanifold $L \in \Phi$, then viewed as a submanifold of M' , is Calibrated by $\text{Re}\varphi'$, which can be described by alternative conditions $\omega^*|_L = 0, \text{Im}\varphi'|_L = 0$. So we can view our moduli-space on M' as Special Lagrangian submanifolds, except for the fact that I' is not an integrable a.c. structure and so M' is only symplectic. If we were able to establish the fibration structure on M by SLag submanifolds of the Calabi-Yau metric instead of ω' , we would have obtained a Calabi-Yau structure on the mirror.

Finally we prove that $\rho : V_a^*/\Lambda_a^* \mapsto V'_a/\Lambda'_a$ is a symplectomorphism and thus M' is a symplectic mirror to M according to Definition 3.3.2. The symplectic structure on V_a^*/Λ_a^* was obtained from the map $\alpha : V_a^* \mapsto T^*\Phi$. Also the symplectic structure on V'_a/Λ'_a was obtained from a map $\xi' : V'_a \mapsto T^*\Phi$. We will prove that

$$\alpha = \xi' \circ \rho$$

and then we are done. This is obviously true outside of bad neighbourhoods. Let now L be in one of bad neighbourhoods, so L has a form $T \times S^1$. Let β^1, β^2 be (an oriented) basis for $H^1(T, \mathbb{Z})$. Then $\beta^1, \beta^2, [dy_3]$ is a basis for $H^1(L, \mathbb{Z})$, which we can view as a basis for Λ_a^* . Let $\beta_1, \beta_2, [S^1]$ be a corresponding dual basis for $H_1(L, \mathbb{Z})$, which we can also view as a basis for Λ_a and Λ'_a . Then $\rho(\beta^1) = -\beta_2, \rho(\beta^2) = \beta_1$ and $\rho([dy_3]) = [S^1]$.

Let $v^i = \xi'(\beta_i)$ and $\gamma = \xi'([S^1])$ in $T^*\Phi$ (because of the product structure γ can be viewed as a 1-form dx_3 on a moduli-space inside our bad neighbourhood). Let v_1, v_2, ∂_{x_3} be a dual basis of $T\Phi$. So if we view v_i as normal vector fields to the fiber then $i_{v_i}\omega^*$ represent cohomology classes β^i . But in a bad neighbourhood $\omega^* = \text{Im}\eta + dx_3 \wedge dy_3$, so $[i_{v_i}\text{Im}\eta] = \beta^i$.

Now by equation 4 we have

$$\alpha(\beta^i)(v_j) = [i_{v_j}\text{Im}\varphi] \cup \beta^i([L]) = [i_{v_j}\text{Im}\eta \wedge -dy_3] \cup \beta^i(L) = \beta^j \cup -[dy_3] \cup \beta^i([L]) = -\beta^i \cup \beta^j([T])$$

Also one easily shows that $\alpha(\beta^i)(\partial_{x_3}) = 0$. So one deduces that $\alpha(\beta^1) = -v^2$, $\alpha(\beta^2) = v^1$ and also $\alpha([dy_3]) = \gamma$. So $\alpha = \xi' \circ \rho$ and we are done.

Remark: It is clear that applying these ideas we can get analogous results for a Calabi-Yau 4-fold N obtained from resolution of a quotient of an 8-torus by \mathbb{Z}_2^3 , there the generators of \mathbb{Z}_2 actions are

$$\begin{aligned} \alpha : z_1, \dots, z_4 &\mapsto -z_1, -z_2, z_3, z_4 \\ \beta : z_1, \dots, z_4 &\mapsto z_1, z_2, -z_3, -z_4 \\ \text{and } \gamma : z_1, \dots, z_4 &\mapsto z_1, -z_2 + 1/2, -z_3 + 1/2, z_4 \end{aligned}$$

Indeed the resolution of the quotient by α and β is a product of 2 K^3 surfaces with a product structure, there each K^3 has a metric Euclidean outside of bad neighbourhoods as before.

For a fixed point set of γ we introduce a bad neighbourhood X in z_2, z_3 coordinates and consider a neighbourhood $Z = T^2 \times X \times T^2$ in T^8 . Then α and β act freely on that neighbourhood. Inside Z we introduces structures as before and this way we get a Kahler metric and a SLAG torus fibration on N .

3.4 Holomorphic functions near SLAG Submanifolds

In this section we examine holomorphic functions in a neighbourhood of a special Lagrangian submanifold. Such examples can be obtained for instance from Calabi-Yau manifold X in $\mathbb{C}P^n$ defined as a zero locus of real polynomials. In that case we have $L = X \cap \mathbb{R}P^n$ a Special Lagrangian submanifold. Let P be some real polynomial of degree k without real roots. Then for any polynomial Q of degree k the function $\frac{Q}{P}$ is a holomorphic function on X in a neighbourhood of L . More generally let L be a fixed point set of an antiholomorphic involution σ and h a meromorphic function on M . Then obviously $\overline{h \circ \sigma}$ is also a meromorphic function on M and so is $g = h \cdot (\overline{h \circ \sigma}) + 1$. Also on L g is real valued and ≥ 1 . So $f = 1/g$ is a holomorphic function in a neighbourhood of L .

An immediate consequence from the fact that SLAG submanifolds are ‘Special’, i.e. $Im\varphi|_L = 0$ if the following

Theorem 3.4.1 : *Let L_0 be Slag Submanifold and f be a holomorphic function in a neighbourhood of L_0 . Let ξ be a function on our moduli-space, $\xi(L) = \int_L f$. Then ξ is a constant function.*

Proof: Consider the following $(n, 0)$ form $\mu = f\varphi$. Then μ is holomorphic, hence closed and obviously $\xi(L) = \int_L \mu$. Q.E.D.

This yields a following corollary :

For $0 < \theta < \pi$ we denote by A_θ an open cone in complex plane given by $(z = re^{i\rho} | r > 0, 0 < \rho < \theta)$.

Corollary 3.4.1 : *Let M be a Calabi-Yau n -fold and f a holomorphic function on some domain U in M . Let $L(t)$ be flow of Slag submanifolds contained in U and $p \in U$ a point s.t. $f(p) = 0$. Suppose that the distance $d(p, L(t)) \rightarrow 0$ as $t \rightarrow \infty$. Then $L(t)$ cannot be contained in the domain $f^{-1}(A_\theta)$ for $\theta < \frac{\pi}{2n}$.*

Remark: This Corollary gives a restriction of how singular SLAG currents might look like .

Proof of Corollary 3.4.1: Suppose $L(t)$ are contained in $W = f^{-1}(A_\theta)$ as above. We can find an $\epsilon > 0$ s.t. $g = f^{n+\epsilon}$ is well defined an holomorphic on W and $g(W) \subset A_{\frac{\pi}{2}}$. Then $h = \frac{g}{2g}$ is holomorphic on W , $h(W) \subset A_{\frac{\pi}{2}}$ and for z close to p we have $|h(z)| \geq const \cdot d(z, p)^{-n-\epsilon}$.

Since $\int_{L(t)} h$ is constant and $Re(h), Im(h) > 0$ on $L(t)$ then $\int_{L(t)} |h|$ is bounded by a constant.

Take now any $\delta > 0$ and pick t and $p_t \in L(t)$ s.t. $d(p, p_t) \leq \delta$. Consider $B = B(p_t, \delta) \cap L(t)$. By Theorem 2.0.1, $vol(B) \geq const \cdot \delta^n$ and on B we have $|h| \geq \frac{const}{\delta^{n+\epsilon}}$. So $\int_{L(t)} |h| \geq \int_B |h| \geq const \cdot \delta^{-\epsilon}$. Now δ was arbitrary - a contradiction. Q.E.D.

Applying these ideas we can also get restriction on SLAG submanifolds in \mathbb{C}^n which are asymptotic to a cone. We have the following theorem:

Theorem 3.4.2 : *Let $L \subset \mathbb{C}^n$ be a special Lagrangian submanifold asymptotic to a cone Λ and let $z_1 \dots z_n$ be coordinates on \mathbb{C}^n . Then L cannot be contained in the cone*

$$B_\theta^\delta = ((z_1, \dots, z_n) | z_1 \in A_\theta, |z_1| > \delta \cdot |z_i|)$$

for $\delta > 0$, $\theta < \pi/2n$.

Remark : The order, to which L is required to be asymptotic to a cone will become clear from the proof.

Proof of the theorem: Consider a flow $L(t)$ of SLAG submanifolds in the unit ball in \mathbb{C}^n with boundary in a unit sphere, $L(t) = (z | t \cdot z \in L, |z| \leq 1)$. We wish to prove that $\int_{L(t)} |z_1|^{-n-\epsilon}$ is uniformly bounded in t for some $\epsilon > 0$ as in the proof of Corollary 3.4.1. This will lead us to a contradiction as before because there are points in $L(t)$ which converge to the origin in \mathbb{C}^n for $t \rightarrow \infty$.

Let d be the distance function to the origin on L . Let $v = \nabla d, w = \frac{v}{||v||^2}$. Then since L is asymptotic to a cone, w will be a well-defined v. field outside some ball B in L and it's length will converge to 1 at ∞ . We will also assume that vector $w(x)$ is close to the line through x and the origin, i.e. that there is a function $g : R_+ \mapsto R_+$ s.t. the length of the orthogonal component of $w(x)$ to this line is $\leq g(||x||)$ and $\int_{[1, \infty)} \frac{g(t)}{t} dt < \infty$.

We extend w inside of B to be a C^∞ v.field on L . Let η_t will be flow of w in time t , then the derivative of d along η_t is 1 outside B .

We can consider the corresponding flow σ_t on L_s ,

$$\sigma_t(x) = \frac{\eta_t(sx)}{s+t}, \quad \sigma_t : L_s \mapsto L_{s+t}$$

Let v_s be a vector field on L_s inducing the flow. One can easily show that on the boundary of L_s we have $||v_s|| \leq const \cdot \frac{g(s)}{s}$.

Pick ϵ small enough so that for $f(z) = z_1^{-n-\epsilon}$ we have Ref, Imf are positive on L_t . Let $h(t) = \int_{L_t} f = \int_{L_t} f\varphi$. We need to prove that $h(t)$ is a bounded function of t and then we are done.

Let Q_t be the boundary of L_t . Then

$$h'(t) = \int_{L_t} \mathcal{L}_{v_t} f \varphi = \int_{L_t} d(i_{v_t} f \varphi) = \int_{Q_t} f \cdot i_{v_t} \varphi$$

Conditions on B_θ^δ imply that $|f|$ is uniformly bounded on Q_t . Also we know that $|v_t| \leq \frac{g(t)}{t}$.

So $|h'(t)| \leq \text{const} \cdot \text{vol}(Q_t) \frac{g(t)}{t}$. Now Q_t converges to the base of the cone, so it's volume is bounded, so $|h'(t)|$ is an integrable function of t by our assumptions, so $h(t)$ is uniformly bounded in t . Q.E.D.

Using those ideas for the 2-torus we get a following fact for analytic functions of 1 complex variable :

Lemma 3.4.1 : *There is no holomorphic function f from the open half disk $D = (re^{i\theta} | 0 < r < 1, 0 < \theta < \pi)$ to itself s.t. $|f(z)| \leq |z|^2$.*

Proof: Consider a domain $D' = (re^{i\theta} | 0 < r < 1, 0 < \theta < \frac{\pi}{2})$.

The map $z \mapsto z^2$ is conformal from D' to D and it is easy to see that is is enough to prove the claim for D' . Also by iterating f we can assume that $|f(z)| \leq |z|^4$.

On a 2-torus $T^2 = \mathbb{R}^2(\text{mod } \mathbb{Z}^2)$ we consider SLag submanifolds

$$L_t = ((x, y) | y = t)$$

We also have a Weierstrass \wp -function, which has a pole of order 2 at the origin.

Consider $L_{\frac{1}{4}}$. We choose a constant c so that $P' = \wp + c$ satisfies $\text{Re}P' , \text{Im}P' > 0$ on $L_{\frac{1}{4}}$. We look at the flow $L_t : t \rightarrow 0$. We have 2 cases :

1) We have a (first) value t_0 s.t. $\text{Re}P'(x, t_0) = 0$ or $\text{Im}P'(x, t_0) = 0$. W.l.o.g. we assume that the second case holds. Let $\text{Re}P'(x, t_0) = a$. Consider $h = \frac{\pi}{2f(\sqrt{P'-a})}$.

Then as we approach t_0 , h remains a holomorphic function with $\text{Re}h, \text{Im}h > 0$ and $|h(z)| > \text{const} \cdot |z - (x + it_0)|^{-2}$. So as in Corollary 3.4.1 we get a contradiction.

2) If $\text{Re}P', \text{Im}P'$ remain positive as $t \rightarrow 0$ then we get a contradiction by looking at $\int_{L_t} |P'|$. Q.E.D.

4 Coassociative Geometry on a G_2 manifold

For an oriented 7-manifold M , let $\bigwedge^3 T^*M$ be a bundle of 3-forms on it. This bundle has an open sub-bundle $\bigwedge_+^3 M$ s.t. $\varphi \in \bigwedge^3 T_p^*M$ is in $\bigwedge_+^3 M$ if there is a linear isomorphism $\sigma : T_p M \mapsto \mathbb{R}^7$ s.t. $\sigma^* \varphi_0 = \varphi$, there φ_0 is a standard G_2 3-form on \mathbb{R}^7 (see [10], p. 294).

A global section φ of $\bigwedge_+^3 M$ defines a topological G_2 structure on M . In particular this defines a Riemannian metric on M . We will be interested in a closed φ . If φ is also co-closed then it is parallel and defines a metric with holonomy contained in G_2 (see [10]). In this case the form $*\varphi$ is a calibration and a calibrated 4-submanifold L is called a coassociative submanifold. This can also be given by an alternative condition $\varphi|_L = 0$. For a closed φ we can define coassociative submanifolds by this condition. They no longer will be Calibrated (because $*\varphi$ is not closed). But nevertheless they are quite interesting because they admit an unobstructed deformation theory. In fact we can copy a proof of theorem 4.5 in [14] to show that their moduli-space Φ is smooth of dimension $b_2^+(L)$ (proof of that theorem in fact never used the fact that $*\varphi$ is closed). If

L is coassociative as before and $p \in L$ we can identify the normal bundle to L at p with self-dual 2-forms on L by a map

$$v \mapsto i_v \varphi$$

for v a normal vector to L . Thus a tangent space to Φ at L can be identified with closed self-dual 2-forms on L .

In a similar way to SLag Geometry we have a following lemma for finite group actions on M :

Lemma 4.0.2 *Suppose a finite group G acts on M preserving φ . Suppose L is a coassociative submanifold, G leaves L invariant and acts trivially on the second cohomology of L . Then G leaves every element of the moduli-space Φ through L invariant. Moreover, suppose $g \in G$ and $x \in M - L$ is an isolated fixed point of g . Then x is not contained in any element in Φ .*

The proof of this lemma is completely analogous to proof of Lemma 3.1.1.

Next we want to point out an example in which a G_2 manifold is a fibration with generic fiber being a coassociative 4-torus. Our manifold M is obtained from resolution of a quotient of a 7-torus by a finite group. We hope to give a systematic way of producing such examples in a future paper.

Let the group be \mathbb{Z}_2^3 with generators

$$\alpha : (x_1, \dots, x_7) \mapsto (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7)$$

$$\beta : (x_1, \dots, x_7) \mapsto (-x_1 + 1/2, 1/2 - x_2, x_3, x_4, -x_5, -x_6, x_7)$$

$$\gamma : (x_1, \dots, x_7) \mapsto (-x_1, x_2, 1/2 - x_3, x_4, -x_5, x_6, -x_7)$$

(compare with [10], p.302). We will follow Joyce's exposition of that example.

The fixed point locus of each generator is a disjoint union of 3-tori. Their fixed loci don't intersect and their compositions have no fixed points. Around each fixed point the quotient looks like $V = T^3 \times B / \pm 1$, there B is a ball in \mathbb{R}^4 . We will show how to get a G_2 structure on resolution of singularities \overline{V} . We will treat a fixed locus of, say, α .

Let x_1, \dots, x_4 be coordinates on \mathbb{R}^4 . Let $\omega_1, \omega_2, \omega_3$ be a standard HyperKahler package on \mathbb{R}^4 . For coordinates x_5, x_6, x_7 , let δ_i be a dual 1-form to x_{8-i} . Then the G_2 3-form on \mathbb{R}^7 is

$$\varphi = \omega_1 \wedge \delta_1 + \omega_2 \wedge \delta_2 + \omega_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3$$

We can use either one of 3 complex structures on \mathbb{R}^4 to identify it with \mathbb{C}^2 . That way for each singularity of the form $\mathbb{R}^4 / \pm 1$ we get a resolution of a singularity \overline{U} and a (non-integrable) HyperKahler package on \overline{U} in a similar way to section 3.3. Suppose, for example, that we used a complex structure I on \mathbb{R}^4 . Then ω_2 and ω_3 lift to \overline{U} . Also ω_1 is replaced by a Kahler form ω'_1 (as in Section 3.3). On $\overline{V} = \overline{U} \times T^3$ we can consider

$$\varphi' = \omega'_1 \wedge \delta_1 + \omega_2 \wedge \delta_2 + \omega_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3$$

and φ' will be a closed G_2 form. Doing so for each fixed locus we get a G_2 structure on M . D. Joyce proved that this 3-form can be deformed to a parallel G_2 form. We will use the form φ' because we can construct a coassociative 4-torus fibration on M for φ' .

On T^7 we have a following coassociative 4-torus fibration

$$T_{a,b,c} = ((x_1, \dots, x_7) | x_1 = a, x_3 = b, x_6 = c)$$

Note that the 4 coordinates on each $T_{a,b,c}$ are chosen so that each generator of \mathbb{Z}_2^3 acts non-trivially on exactly 2 of those coordinates. Those $T_{a,b,c}$ become coassociative tori on M and fill a big open neighbourhood of M . We would like to see what happens then these tori enter a 'bad' neighbourhood V above. For that we have to consider a bigger tubular neighbourhood $W = X \times T^3$, there

$$X = (v \in T^4 | v = w + (0, a, 0, b), \text{ there } w \in U \text{ and } a, b \in \mathbb{R}/\mathbb{Z})$$

So W are exactly those points in T^7 which are contained on $T_{a,b,c}$ for a torus $T_{a,b,c}$, which intersects V . We have a resolution of singularities \overline{W} , which can be viewed as a neighbourhood in M .

We would like to investigate coassociative tori in \overline{W} . As we mentioned, we have 3 different ways to resolve a singularity using either one of the structures I, J, K . We have 2 different cases :

1) The structure we are using is either I or K . We can assume that it is I . The G_2 form looks like

$$\omega'_1 \wedge \delta_1 + \omega_2 \wedge \delta_2 + \omega_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3$$

Our tori will look like

$$T_c \times L$$

there T_c is a torus in x_5, x_6, x_7 coordinates defined by condition $x_6 = c$ and L is in \overline{U} defined by conditions: $\omega'_1|_L = 0, \omega_3|_L = 0$, i.e it is a Special Lagrangian torus as in section 3.3. The results of section 3.3 precisely apply to show that the compactified moduli-space fibers the neighbourhood \overline{W} .

2) The structure we are using is J . Then we have a package $\omega_1, \omega'_2, \omega_3$ and the G_2 form looks correspondingly. Then our tori again look like $T_c \times L$, there L satisfies $\omega_1|_L = 0, \omega_3|_L = 0$, i.e. they are holomorphic tori with respect to a structure J . So what we get is a neighbourhood in K^3 with a standard holomorphic fibration over a neighbourhood in S^2 . So in any case our manifold M fibers with generic fiber being a coassociative 4-torus.

Finally we want to construct a topological mirror for this torus fibration. We will use definitions from section 3.3. It is clear that because of the local product structure the local monodromy representation is isomorphic to the dual representation. Hence we can construct a dual fibration by performing a surgery for each 'bad' neighbourhood \overline{W} in a similar way to section 3.3. So for instance if \overline{W} is a neighbourhood of α then we glue \overline{W} to $M - \overline{W}$ along a boundary by a map

$$\eta : (x_1, \dots, x_7) \mapsto (x_1, -x_4, x_3, x_2, x_5, x_6, x_7)$$

Also near the boundary of \overline{W} we have $\eta^*(\omega_1) = \omega_3$, $\eta^*(\omega_3) = -\omega_1$, $\eta^*(\omega_2) = \omega_2$. So we can glue a closed G_2 form $\varphi^* = \omega_3 \wedge \delta_1 + \omega_2 \wedge \delta_2 - \omega_1 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3$ inside of \overline{W} to φ' outside of \overline{W} to get a closed G_2 form φ'' on the mirror. Also the mirror fibration is a fibration by coassociative 4-tori with respect to φ'' .

Remark: The original example in Joyce's paper [10] was a quotient by a slightly different \mathbb{Z}_2^3 action with generators

$$\alpha : x_1, \dots, x_7 \mapsto -x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7$$

$$\beta : x_1, \dots, x_7 \mapsto -x_1, 1/2 - x_2, x_3, x_4, -x_5, -x_6, x_7$$

$$\gamma : x_1, \dots, x_7 \mapsto 1/2 - x_1, x_2, 1/2 - x_3, x_4, -x_5, x_6, -x_7$$

We get a closed G_2 form φ on the resolution of singularities similarly to previous example. Our manifold M again will be fibered by coassociative 4-tori if we start from a family $T_{a,b,c} = ((x_1, \dots, x_7) | x_1 = a, x_2 = b, x_7 = c)$. Indeed we consider a neighbourhood U_i of a fixed component of one of the generators and a bigger neighbourhood $X_i = (v \in T^7 | v = u + (0, 0, a_1, a_2, a_3, a_4, 0) \text{ s.t. } u \in U \text{ and } a_i \in \mathbb{R}/\mathbb{Z})$. Then X_i are disjoint, so we can use product structure on X_i to get a coassociative torus fibration on M as in the previous example.

References

- [1] M. Anderson : Complete minimal varieties in Hyperbolic space, Inven. Math. 69 (1982), no.3, 477-494
- [2] R. Bott and L. Tu : Differential forms in Algebraic Topology, Springer Graduate texts in mathematics.
- [3] R. Bryant : Some examples of Special Lagrangian Tori, math.DG/99020/6v2.
- [4] M. Gross : Topological Mirror Symmetry, math.AG/9909015
- [5] M. Gross : Special Lagrangian Fibrations 2 - Geometry, alg-geom 9809072
- [6] R. Harvey : Spinors and Calibrations, Perspectives in Mathematics, vol 9
- [7] R. Harvey and H. B. Lawson : Calibrated Geometries, Acta Math. 148 (1982).
- [8] N. Hitchin : The moduli-space of special Lagrangian submanifolds , dg-ga/9711002.
- [9] D. Joyce : On the topology of desingularization of Calabi-Yau orbifolds, math.AG/9806146.
- [10] D. Joyce : Compact Riemannian 7-manifolds with holonomy G_2 I and II, J. Diff. Geom. 43 (1996), 291-328, 329-375

- [11] I. Kim : Relative isoperimetric inequality and linear isoperimetric inequality for minimal submanifolds. *Manuscripta Math.* 97 (1998) no.3, 343-352
- [12] P. Kronheimer : The construction of ALE spaces as HyperKahler quotients, *Jour. Diff. Geom.* 29 (1989).
- [13] P. Lu : Special Lagrangian tori on a Borcea-Voisin threefold, *math.DG/9902063*
- [14] R. C. McLean : Deformations of Calibrated submanifolds, *Comm. Anal. Geom.* 6(1998) 705-747
- [15] D. McDuff and D. Salomon : J-Holomorphic Curves and Quantum Cohomology, University Lecture series, vol. 6
- [16] W.H. Meeks, S.T. Yau : Topology of three dimensional manifolds and the embeddings problems in minimal surface theory, *Annals of Math.* 112 (1980) 441-484
- [17] M. Micallef, B. White : The structure of branch points in minimal surfaces and in pseudoholomorphic curves, *Ann. of Math.*(2) 141 (1995), no.1 , 35-85
- [18] F. Morgan : Geometric Measure Theory, A Beginner's Guide, Academic Press.
- [19] S. Salamon : Riemannian Geometry and Holonomy Groups, Pitman Press.
- [20] S.Salur : Deformations of Special Lagrangian submanifolds, *math.DG/9906048*
- [21] Y. Ruan and G. Tian : A mathematical theory of Quantum Cohomology, *Jour. Diff. Geom.* vol 42, no 2 (1995).
- [22] A. Strominger, S. T. Yau and E. Zaslow : Mirror Symmetry is T-Duality, *Nucl. Phys.* B476(1996).
- [23] O. Viro: Construction of M-Surfaces, *Func. Anal. Appl.* 13(3) 1979
- [24] J. Wolfson : Gromov's compactness of pseudoholomorphic curves and symplectic geometry, *Jour. Diff. Geom.* 28 (1988).
- [25] S. T. Yau : On the Ricci curvature of compact Kahler manifold and the complex Monge-Ampere equation 1, *Comm. Pure Appl. Math* 31 (1978).

Massachusetts Institute of Technology
E-mail : egold@math.mit.edu